

Killing Vector Fields in Three Dimensions: A Method to Solve Massive Gravity Field Equations

Metin Gürses

Department of Mathematics, Faculty of Sciences
Bilkent University, 06800 Ankara - Turkey

July 30, 2010

Abstract

Killing vector fields in three dimensions play important role in the construction of the related spacetime geometry. In this work we show that when a three dimensional geometry admits a Killing vector field then the Ricci tensor of the geometry is determined in terms of the Killing vector field and its scalars. In this way we can generate all products and covariant derivatives at any order of the ricci tensor. Using this property we give ways of solving the field equations of Topologically Massive Gravity (TMG) and New Massive Gravity (NMG) introduced recently. In particular when the scalars of the Killing vector field (timelike, spacelike and null cases) are constants then all three dimensional symmetric tensors of the geometry, the ricci and einstein tensors, their covariant derivatives at all orders, their products of all orders are completely determined by the Killing vector field and the metric. Hence the corresponding three dimensional metrics are strong candidates of solving all higher derivative gravitational field equations in three dimensions.

1 Introduction

Einstein gravity in three dimensions is trivial because it contains no dynamics. A theory in three dimensions with dynamics is known as topologically massive gravity (TMG) [1],[2]. The action of TMG is the sum of the standard Einstein-Hilbert and Chern-Simons terms in three dimensions. TMG with cosmological constant is considered recently [3], [4]. This theory violates parity, it is renormalizable but not unitary. Recently a new massive gravity theory (NMG) is introduced. In the framework of perturbation theory NMG is both renormalizable and unitary [5]-[7] (see also [8], [9]). Although higher derivative theories play important role to have a consistent quantum gravity it is however very difficult to obtain classical solutions of these theories. For this purpose there have been several attempts to solve these theories, in particular to find black hole solutions. In these attempts different methods were used. In [10] we used perfect fluid solutions of Einstein theory to solve the TMG field equations. In [11] - [16] by assuming commuting two Killing vector fields Clement gave some solutions of the TMG and NMG field equations. Another method is the principle of symmetric criticality [17], [18], [19]. In [20] algebraic classification of the Ricci tensor is used and reviewed all known solutions of the TMG. In this work it has been also emphasized the importance of the Killing vector fields in obtaining the solutions of the TMG field equations. It was shown that (see also [21]) all solutions are locally belong to one of the three solutions: Timelike-squashed AdS_3 , Spacelike-squashed AdS_3 and AdS pp-waves [22], [23], [24]. In [25] Chow et al used the Kundt metrics to solve TMG field equations and they obtained new solutions of the TMG. Another method is to use the Gödel type metrics [26]-[28] in three dimensions [29]. There are also some other attempts [30]-[32] to solve the field equations of TMG. AdS pp-wave solutions of NMG field equations were considered in [33]. The importance of the Killing vectors in finding the solutions of the massive gravity field equations in three dimensions is also observed in some of these works. If they exist, Killing vectors in three dimensions are not only important in the derivation of the exact solutions but they control the whole geometry of spacetime.

In this work we shall introduce a new method for three dimensional gravity theories. The starting point is to assume a one parameter family of isometry. This implies existence of a Killing vector field in the spacetime geometry. In general this assumption may simplify and reduce the number the

field equations by choosing a suitable coordinate patch so that one coordinate in the line element becomes cyclic. In the case of four and higher dimensional geometries this may give information about some components of the curvature and hence the ricci tensors but it does not provide all components of these tensors. We shall show that this is possible in three dimensions.

If a three dimensional spacetime admits a non-null Killing vector field, writing it in terms of two scalars, its norm and rotation scalar, we show that the metric and the ricci tensors are completely determined in terms of the Killing vector, its scalars and vectors obtained through these scalars (Theorem 1). Physical interpretation of the corresponding spacetime metric is that it solves the Einstein-Maxwell- Dilaton field equations in three dimensions. When the scalars of the Killing vector field become constant then the corresponding ricci tensor becomes relatively simple (Theorem 2). We show that the spacetime metric, in this case, solve the field equations of TMG and NMG. Indeed these metrics (for timelike and spacelike cases) solve not only these field equations, but they solve all higher (curvature) derivative theories at any order. When the Killing vector field is null we have similar result (Theorem 3). Without referring to the metric tensor we construct the Einstein tensor when the spacetime geometry admits a null vector. The corresponding metric solves Einstein field equations with a null fluid distribution and an exotic scalar field. When rotation scalar of the null Killing vector field become constant the Einstein tensor becomes more simpler (Theorem 4) and the metric solves the field equations of TMG and NMG. Both for non-null and null cases when the scalars of the Killing vector fields become constant then the metric solves not only the field equations of TMG and NMG but solves also all higher derivative gravitational field equations in three dimensions .

In Section 2 we give the ricci and metric tensors in terms of the non-null Killing vector field and its scalars. The Einstein tensor represents a charged fluid with a dilaton field. In particular if the scalars are constants the ricci tensor is expressed in terms of the Killing vector field and the Gaussian curvature of the two dimensional background space in Section 3. In Section 4 we show that when the Killing vector field is null then the ricci tensor is calculated in terms of the null Killing vector field and its rotation scalar. In Section 5 we investigate the case when the rotation scalar is a constant. In Sections 6 and 7, by using the ricci tensors found in the previous Sections, we solve the field equations of the TMG and NMG respectively.

2 Non-Null Killing Vectors in Three Dimensions

In this section we shall construct the metric and Einstein tensors in terms of scalar functions of a non-null Killing vector. Our conventions in this paper are similar to the convention of Hawking-Ellis [34].

Let u^μ be a non-null Killing vector field satisfying the Killing equation

$$u_{\mu;\nu} + u_{\nu;\mu} = 0 \quad (1)$$

with $u_\alpha u^\alpha = \alpha \neq 0$. One can show that

$$u^\alpha u_{\alpha;\mu} = \frac{1}{2} \alpha_{,\mu}, \quad u^\alpha u_{\mu;\alpha} = -\frac{1}{2} \alpha_{,\mu} \quad (2)$$

The norm function α is in general not a constant. It is possible to write the Killing equation as

$$u_{\mu;\nu} = \frac{1}{2} (u_{\mu;\nu} - u_{\nu;\mu}) = \eta_{\mu\nu\alpha} v^\alpha \quad (3)$$

where v^α is any vector field and $\eta_{\mu\nu\alpha}$ is the Levi-Civita alternating symbol. Using (2) we find that

$$v^\rho = w u^\rho + \eta^{\rho\mu\sigma} \phi_{,\mu} u_\sigma \quad (4)$$

where $w = \frac{u_\mu v^\mu}{\alpha}$. Using (4) in (3) we get

$$u_{\mu;\nu} = w \eta_{\mu\nu\alpha} u^\alpha + u_\mu \phi_{,\nu} - u_\nu \phi_{,\mu} \quad (5)$$

where $\phi = \frac{1}{2} \ln \alpha$. The functions w and ϕ specify the Killing vector field and they are assumed to be linearly independent. Differentiating (5) with respect to x^σ and anti-symmetrizing both sides with respect ν and σ we get

$$\begin{aligned} u_{\mu;\nu\sigma} - u_{\mu;\sigma\nu} &= w_{,\sigma} \eta_{\mu\nu\alpha} u^\alpha - w_{,\nu} \eta_{\mu\sigma\alpha} u^\alpha - w \eta_{\mu\nu\alpha} \phi^\alpha_{,\sigma} u_\sigma + w \eta_{\mu\sigma\alpha} \phi^\alpha_{,\nu} u_\nu \\ &\quad - 2 \eta_{\nu\sigma\alpha} u^\alpha \phi_{,\mu} - u_\nu \phi_{,\sigma} \phi_{,\mu} + u_\sigma \phi_{,\mu} \phi_{,\nu} - u_\nu \phi_{,\mu\sigma} + u_\sigma \phi_{,\mu\nu} \\ &\quad + w^2 (g_{\mu\nu} u_\sigma - g_{\mu\sigma} u_\nu) \end{aligned} \quad (6)$$

Using the identities

$$w_{,\sigma} \eta_{\mu\nu\alpha} u^\alpha + w_{,\mu} \eta_{\nu\sigma\alpha} u^\alpha + w_{,\nu} \eta_{\sigma\mu\alpha} u^\alpha = \eta_{\sigma\mu\nu} u^\alpha \phi_{,\alpha} = 0, \quad (7)$$

$$\eta_{\mu\nu\alpha} \phi^\alpha u_{,\sigma} + \eta_{\nu\sigma\alpha} \phi^\alpha u_{,\mu} + \eta_{\sigma\mu\alpha} \phi^\alpha u_{,\nu} = \eta_{\sigma\mu\nu} \phi^\alpha u_\alpha = 0 \quad (8)$$

then Eq. (6) reduces to (Here $\phi^\alpha = g^{\alpha\mu} \phi_{,\mu}$)

$$\begin{aligned} u_{\mu;\nu\sigma} - u_{\mu;\sigma\nu} &= w \eta_{\nu\sigma\alpha} \phi^\alpha u_\mu - \eta_{\nu\sigma\alpha} u^\alpha y_\mu + \phi_{,\mu} (u_\sigma \phi_{,\nu} - u_\nu \phi_{,\sigma}) \\ &\quad + w^2 (g_{\mu\nu} u_\sigma - g_{\mu\sigma} u_\nu) \end{aligned} \quad (9)$$

where

$$y_\mu = w_{,\mu} + 2w \phi_{,\mu} \quad (10)$$

Using the ricci identity $u_{\mu;\nu\sigma} - u_{\mu;\sigma\nu} = R^\rho{}_{\mu\nu\sigma} u_\rho$ in (9) we get

$$\begin{aligned} R^\rho{}_{\mu\nu\sigma} u_\rho &= w \eta_{\nu\sigma\alpha} \phi^\alpha u_\mu - \eta_{\nu\sigma\alpha} u^\alpha y_\mu + \phi_{,\mu} (u_\sigma \phi_{,\nu} - u_\nu \phi_{,\sigma}) \\ &\quad + w^2 (g_{\mu\nu} u_\sigma - g_{\mu\sigma} u_\nu) \end{aligned} \quad (11)$$

and hence we obtain

$$R^\rho{}_{\nu} u_\rho = -\zeta_\nu - (\nabla^2 \phi + 2w^2) u_\nu \quad (12)$$

where

$$\zeta_\mu = \eta_{\mu\alpha\beta} y^\alpha u^\beta \quad (13)$$

In three dimensions the Weyl tensor vanishes and hence curvature tensor is expressed solely in terms of the ricci tensor

$$R_{\rho\mu\nu\alpha} = -R_{\mu\nu} g_{\rho\alpha} + R_{\rho\nu} g_{\mu\alpha} + R_{\mu\alpha} g_{\rho\nu} - R_{\rho\alpha} g_{\mu\nu} + \frac{R}{2} (g_{\rho\alpha} g_{\mu\nu} - g_{\rho\nu} g_{\mu\alpha}) \quad (14)$$

Combining equations (11) and (14) we obtain

$$\bar{R}_{\mu\nu} u_\sigma - \bar{R}_{\mu\sigma} u_\nu + \zeta_\sigma g_{\mu\nu} - \zeta_\nu g_{\mu\sigma} = -\eta_{\nu\sigma\alpha} u^\alpha y_\mu + w \eta_{\nu\sigma\alpha} \phi^\alpha u_\mu, \quad (15)$$

where

$$\bar{R}_{\mu\nu} = -R_{\mu\nu} + (P + \frac{R}{2} - w^2) g_{\mu\nu} - \phi_{;\mu\nu} - \phi_{,\mu} \phi_{,\nu}, \quad (16)$$

$$P = \nabla^2 \phi + 2\phi_{,\alpha} \phi^\alpha + 2w^2 \quad (17)$$

In addition to the identities (7) we have also the following identities

$$z_\mu u_\sigma - z_\sigma u_\nu = \alpha \eta_{\mu\sigma\rho} \phi^\rho, \quad (18)$$

$$\zeta_\mu y_\sigma - \zeta_\sigma y_\nu = -(y_\alpha y^\alpha) \eta_{\mu\sigma\rho} u^\rho \quad (19)$$

Using these identities in (20) we arrive at the following theorem:

Theorem 1. *If a three dimensional spacetime admits a non-null Killing vector field u^μ then its metric and the Einstein tensors are respectively given by*

$$g_{\mu\nu} = \frac{1}{\alpha} u_\mu u_\nu + \frac{1}{\zeta^\alpha \zeta_\alpha} \zeta_\mu \zeta_\nu + \frac{1}{y_\alpha y^\alpha} y_\mu y_\nu, \quad (20)$$

$$G_{\mu\nu} = -\frac{H}{\alpha} u_\mu u_\nu + \frac{1}{\alpha} [(w z_\mu - \zeta_\mu) u_\nu + (w z_\nu - \zeta_\nu) u_\mu] \quad (21)$$

$$+ (\nabla^2 \phi + w^2) g_{\mu\nu} - \phi_{,\mu} \phi_{,\nu} - \phi_{;\mu\nu}, \quad (22)$$

$$H = 2\nabla^2 \phi + 3w^2 + \frac{1}{2} R - \phi^\alpha \phi_{,\alpha}, \quad (23)$$

$$z_\mu = \eta_{\mu\alpha\beta} \phi^\alpha u^\beta \quad (24)$$

where R is the ricci scalar provided that the functions w and ϕ are not constant at the same time. Here we assume that α nowhere vanishes in the spacetime.

Remark 1. When u^μ is a timelike Killing vector field, letting $\alpha = -e^{2\phi}$, then it can be shown that the source of the Einstein equations is composed of a charged fluid distribution and a dilaton field ϕ (in Einstein frame). The electromagnetic field tensor $f_{\mu\nu}$ is given by

$$f_{\mu\nu} = w \eta_{\mu\nu\alpha} u^\alpha - u_\mu \phi_{,\nu} + u_\nu \phi_{,\mu} \quad (25)$$

so that

$$f^{\mu\nu}{}_{;\mu} = -\zeta^\nu + (\nabla^2 \phi - 2w^2) u^\nu \quad (26)$$

The Maxwell energy momentum tensor is given as

$$\begin{aligned} M_{\mu\nu} = & -\frac{1}{2}\alpha(w^2 + \phi_{,\alpha}\phi^\alpha)g_{\mu\nu} + (w^2 + \phi_{,\alpha}\phi^\alpha)u_\mu u_\nu \\ & + \alpha\phi_{,\mu}\phi_{,\nu} - w(z_\mu u_\nu + z_\nu u_\mu) \end{aligned} \quad (27)$$

Then field equations are

$$\begin{aligned} G_{\mu\nu} = & \frac{1}{4}e^{-2\phi}M_{\mu\nu} + e^{-2\phi}(-2\nabla^2\phi + 4\phi_{,\alpha}\phi^\alpha + e^{-2\phi}f^2 - \frac{R}{2})u_\mu u_\nu \\ & + (\nabla^2\phi + \frac{1}{4}e^{-2\phi}f^2)g_{\mu\nu} - e^{-2\phi}(J_\mu u_\nu + J_\nu u_\mu) - \phi_{;\mu\nu} \end{aligned} \quad (28)$$

where $J^\mu = f^{\alpha\mu}{}_{;\alpha}$ and $f^2 = f_{\alpha\beta}f^{\alpha\beta}$.

All our analysis so far is valid when the norm α of the Killing vector field u^μ is different than zero everywhere in spacetime. If there are points where $\alpha = 0$ we exclude them from the spacetime manifold for causality reasons. This set of points define the Killing horizons which are in general coordinate singularities of the spacetime [34]. We may find suitable coordinate patches where these coordinate singularities are removed (maximal extension) [35]. Examples to such Killing horizons in three dimensions are discussed in the BTZ spacetime [36]. Killing horizon, in general, is a subset of the set of fixed points of the isometry whose generator is the Killing vector u . Such a set is called local isometry horizon (LIH) by Carter [35]. Fixed points of an isometry are the set of points where the Killing vector field vanish, i.e., critical points of u , $u(p) = 0$. It is obvious that our analysis fails at the points of LIH. For a given Killing vector field we have [37]

$$u_{\mu;\alpha\beta} = u_\rho R^\rho{}_{\beta\mu\alpha} \quad (29)$$

At the critical points of u the second covariant derivative of the Killing vector field vanishes. This implies that all higher derivatives of u at p are either zero or expressed in terms of the first partial derivative $F_{\mu\nu}(p) = u_{\mu,\nu}(p)$. The integrability of (29) at p gives

$$F_{\mu\alpha}(p) R^\alpha{}_\nu + F_{\nu\alpha}(p) R^\alpha{}_\mu = 0 \quad (30)$$

which imply that the Segre type of the ricci tensor is special at the critical points of u [38], [39]. In a neighborhood U of the critical point p the Killing vector has the linearized form

$$u_\mu = F_{\mu\alpha}(p) Z^\alpha, \quad (31)$$

where $Z^\mu = x^\mu - p^\mu$. Theorem 1 is valid in U with

$$\alpha = F^\mu{}_\alpha(p) F_{\mu\beta}(p) Z^\alpha Z^\beta, \quad w = -\frac{1}{2\alpha} \eta^{\mu\nu\alpha} F_{\mu\nu}(p) F_{\alpha\beta}(p) Z^\beta. \quad (32)$$

To identify the real structure of the spacetime about the critical points for a given particular exact solution it may be useful to study the maximal extension of the corresponding coordinate patches.

If the norm α of a Killing vector field is constant everywhere in spacetime then there are no fixed points of the corresponding isometry. Then Theorem 1 is valid everywhere in spacetime. This is the case which will be considered in the next sections. This will be true not only for spacelike and timelike Killing vector fields but true also for the null Killing vector fields.

3 Non-null Case With Constant Scalars

In the previous section we assumed that the scalars w and ϕ of the Killing vector field are not simultaneously constants. If w and ϕ are both constants then we get

$$u_{\mu;\nu} = w \eta_{\mu\nu\alpha} u^\alpha \quad (33)$$

Then following the above procedure and using (21) we have the following result.

Theorem 2. *If a three dimensional spacetime admits a non-null Killing vector field u^μ with w and α are both constants then the Einstein tensor is given as*

$$G_{\mu\nu} = -\frac{1}{\alpha} (3w^2 + \frac{1}{2} R) u_\mu u_\nu + w^2 g_{\mu\nu} \quad (34)$$

In this case, since the vector ζ_μ vanishes in (15), the metric tensor is not obtained.

Remark 2. If u^μ is a timelike Killing vector field the above Einstein tensor represents a dust distribution in a spacetime with a positive cosmological constant.

For completeness let us give the spacetime metrics of each case: We shall consider the case where α and w are constants.

1. Let $u^\mu = -\frac{1}{u_0} \delta_0^\mu$ be the timelike vector field then $u^\mu u_\mu = -1$. Hence $\alpha = -1$ and $u_\mu = g_{\mu\alpha} u^\alpha$, then $g_{00} = -u_0^2, g_{01} = -u_0 u_1, g_{02} = -u_0 u_2$. Then the spacetime metric can be taken as $(x^\mu = (t, r, z))$

$$ds^2 = -(u_0 dt + u_1 dr + u_2 dz)^2 + M^2 dr^2 + 2L dr dz + N^2 dz^2 \quad (35)$$

where $M^2 = g_{11} + u_1^2, N^2 = g_{22} + u_2^2, L = g_{12} + u_1 u_2$. Here u_0 is a constant (due to the Killing equation) and the metric functions M, N, L, u_1 and u_2 depend on the variables r and z . The metric in (35) is of Gödel type which was used in [29]. The Einstein tensor is given in (34) with $\alpha = -1$. The only field equation is

$$u_{2,r} = u_{1,z} + 2w \Delta / u_0, \quad (36)$$

where $\Delta = |u_0| \sqrt{M^2 N^2 - L^2}$. For simplicity consider the case $L = 0$. Then $R = 2K + 2w^2$ where K is the Gaussian curvature of the locally Euclidian two dimensional background space with the line element $ds_2^2 = M^2 dr^2 + N^2 dz^2$. Metric (35) gives the Einstein tensor (34) where the only field equation is given in equation (36) with $\alpha = -1$ and w is constant.

2. Let $u^\mu = \frac{1}{u_2} \delta_2^\mu$ be the spacelike vector field then $u^\mu u_\mu = 1$. Hence $\alpha = 1$ and $u_\mu = g_{\mu\alpha} u^\alpha = \frac{1}{u_2} g_{\mu 2}$, then $g_{22} = u_2^2, g_{12} = u_1 u_2, g_{02} = u_0 u_2$. Then spacetime metric can be taken as

$$ds^2 = (u_2 dz + u_1 dr + u_0 dt)^2 + M^2 dr^2 + 2L dr dt - N^2 dt^2, \quad (37)$$

where $M^2 = g_{11} - u_1^2$, $N^2 = u_0^2 - g_{00}$, $L = g_{01} - u_0 u_1$, u_2 is a constant (due to the Killing equation) and the metric functions M, N, L, u_1 and u_0 depend on the variables t and r . The Einstein tensor is given in (34) with $\alpha = 1$. The only field equation is

$$u_{0,r} = u_{1,t} + 2w \Delta / u_2, \quad (38)$$

where $\Delta = |u_2| \sqrt{M^2 N^2 + L^2}$. In both cases the ricci scalar is R is not constant. In this case as well, take $L = 0$ for simplicity then we have $R = 2K + 2w^2$ where K is the Gaussian curvature of two dimensional locally minkowskian background space with the line element $ds_2^2 = M^2 dr^2 - N^2 dt^2$. Metric (37) gives the Einstein tensor (34) where the only field equation is given in (38) with $\alpha = 1$ and w constant. In both cases the ricci and Einstein tensors are respectively given as

$$R_{\mu\nu} = -\frac{1}{\alpha} (4w^2 + K) u_\mu u_\nu + (2w^2 + K) g_{\mu\nu}, \quad (39)$$

$$G_{\mu\nu} = -\frac{1}{\alpha} (4w^2 + K) u_\mu u_\nu + w^2 g_{\mu\nu}. \quad (40)$$

It is possible to pass from timelike case to spacelike case by complex transformation [13]. When the background two dimensional spaces are de-sitter or anti de-sitter, in each case, these solutions are known as timelike squashed AdS_3 and spacelike squashed AdS_3 respectively [20],[21], [11].

When R is a constant, the Cotton tensor and the Laplacian of the ricci tensor take simple forms (for both cases). They will used later.

$$C_{\mu\nu} = -w (4w^2 + K) [g_{\mu\nu} - \frac{3}{\alpha} u_\mu u_\nu], \quad (41)$$

$$\nabla^2 R_{\mu\nu} = -2w^2 (4w^2 + K) [g_{\mu\nu} - \frac{3}{\alpha} u_\mu u_\nu] \quad (42)$$

4 Three Dimensional Spacetime Admitting a Null Killing Vector Field

In this section we shall construct the Einstein tensor with respect to the scalar functions of a null Killing vector field.

Let ξ_μ be Killing Vector field. It satisfies the Killing equation

$$\xi_{\mu;\nu} = \frac{1}{2}(\xi_{\mu;\nu} - \xi_{\nu;\mu}) = \eta_{\mu\nu\alpha} v^\alpha \quad (43)$$

where $\eta_{\mu\nu\alpha} = \sqrt{-\det(g)} \epsilon_{\mu\nu\alpha}$ and v^μ is an arbitrary vector field and a semi-colon defines the covariant derivative with respect to the metric $g_{\mu\nu}$. Here the $\epsilon_{\mu\nu\alpha}$ is the three dimensional Levi-Civita alternating symbol. Since $\xi^\alpha \xi_{\alpha;\beta} = 0$ and $\xi^\alpha \xi_{\beta;\alpha} = 0$, then $v^\mu = w \xi^\mu$. Hence (43) becomes

$$\xi_{\mu;\nu} = w \eta_{\mu\nu\alpha} \xi^\alpha \quad (44)$$

where w is an arbitrary function. Taking one more covariant derivative of (44) we find that

$$\xi_{\mu;\nu\alpha} = w_{,\alpha} \eta_{\mu\nu\rho} \xi^\rho - w^2 (g_{\mu\alpha} \xi_\nu - g_{\nu\alpha} \xi_\mu) \quad (45)$$

Using this equation and the Ricci identity we get

$$\begin{aligned} \xi_{\mu;\nu\alpha} - \xi_{\mu;\alpha\nu} &= w_{,\alpha} \eta_{\mu\nu\rho} \xi^\rho - w_{,\nu} \eta_{\mu\alpha\rho} \xi^\rho - w^2 (g_{\mu\alpha} \xi_\nu - g_{\nu\mu} \xi_\alpha) \\ &= R^\rho{}_{\mu\nu\alpha} \xi_\rho. \end{aligned} \quad (46)$$

On the other hand, since the Weyl tensor vanishes in three dimensions the Riemann tensor is expressed totally in terms of the Ricci tensor as expressed in (14). Using (46) and (14) we find

$$Q_{\mu\nu} \xi_\alpha - Q_{\mu\alpha} \xi_\nu - \zeta_\nu g_{\mu\alpha} + \zeta_\alpha g_{\mu\nu} = w_{,\alpha} \eta_{\mu\nu\rho} \xi^\rho - w_{,\nu} \eta_{\mu\alpha\rho} \xi^\rho \quad (47)$$

where

$$\zeta_\mu = \eta_{\mu\alpha\beta} w^\alpha \xi^\beta, \quad (48)$$

$$Q_{\mu\nu} = -R_{\mu\nu} + \left(w^2 + \frac{R}{2}\right) g_{\mu\nu}. \quad (49)$$

Here R is the Ricci scalar and $w^\alpha = g^{\alpha\mu} w_{,\mu}$. Let a vector X_μ be defined as

$$X_\mu = \eta_{\mu\alpha\beta} F^\alpha \xi^\beta \quad (50)$$

where $F^\mu = g^{\mu\alpha} F_{,\alpha}$ and F is any function independent of t . It is easy to show that $\xi^\mu X_\mu = X^\mu X_\mu = 0$. Since the orthogonal null vectors can only be parallel then we have

$$X_\mu = \sigma \xi_\mu, \quad (51)$$

Hence

$$\zeta_\mu = \psi \xi_\mu, \quad (52)$$

and

$$\xi_\mu w_{,\nu} - \xi_\nu w_{,\mu} = \psi \eta_{\mu\nu} \xi^\alpha, \quad \psi \neq 0 \quad (53)$$

which leads also

$$w^\alpha w_{,\alpha} = \psi^2. \quad (54)$$

Using (53) in (47) we obtain

$$Q_{\mu\nu} + \psi g_{\mu\nu} - \frac{1}{\psi} w_{,u} w_{,\nu} + \rho \xi_\mu \xi_\nu = 0. \quad (55)$$

Hence

$$R_{\mu\nu} = \rho \xi_\mu \xi_\nu - (\psi + 2w^2) g_{\mu\nu} - \frac{1}{\psi} w_{,u} w_{,\nu} \quad (56)$$

where ρ is a function to be determined and the ricci scalar is given by

$$R = -6w^2 - 4\psi. \quad (57)$$

Then we have the following theorem:

Theorem 3. *Let ξ^μ be a null Killing vector field of a three dimensional spacetime, then the corresponding Einstein tensor is given by*

$$G_{\mu\nu} = \rho \xi_\mu \xi_\nu - \frac{1}{\psi} w_{,u} w_{,\nu} + (w^2 + \psi) g_{\mu\nu} \quad (58)$$

where ρ is a function to be determined. Furthermore the scalar field satisfies the following partial differential equation

$$\nabla_\mu \left(\frac{1}{\psi} w^\mu \right) = 2w. \quad (59)$$

It is quite interesting that by imposing a null Killing vector in a three dimensional spacetime geometry we obtain the form of the Einstein tensor without knowing the metric tensor explicitly, except the function ρ . The source of the Einstein field equations is composed of a null fluid and a scalar field. The scalar field satisfies a nonlinear differential equation (59). From the Einstein equations the energy momentum tensor $T_{\mu\nu}$ of the source satisfies $T_{\mu\nu} \xi^\mu \xi^\nu = 0$. For any timelike unit vector field u^μ we have $T_{\mu\nu} u^\mu u^\nu = \rho (\xi^\mu u_\mu)^2 - \frac{1}{\psi} (w_{,\mu} u^\mu)^2 + (\psi + w^2)$. Choosing $\psi = -\sqrt{w_{,\mu} w^\mu}$ then the sign of the energy depends on the sign of the function ρ .

As a summary we can conclude in this section as: If ξ_μ is a null Killing vector field of a three dimensional spacetime geometry then the corresponding metric solves the Einstein-null fluid-scalar field equations. The only field equations are (44) and the scalar field equation given in Eq.(59).

5 Null Case Without Scalar Field

In the previous section we considered the case $\psi \neq 0$. Vanishing of ψ is a distinct case. Due to this reason we consider it in a separate section. We have the following result:

Theorem 4. *When the scalar function ψ in Theorem 3 is zero then w becomes a constant and the ricci, Einstein tensors take simple forms*

$$R_{\mu\nu} = \rho \xi_\mu \xi_\nu - 2w^2 g_{\mu\nu}, \quad (60)$$

$$G_{\mu\nu} = \rho \xi_\mu \xi_\nu + w^2 g_{\mu\nu}, \quad (61)$$

with $R = -6w^2$.

For explicit solutions we shall adopt some simplifying assumptions.

(5.A). As an example let us assume that $\xi^\mu = \delta_0^\mu$. Then $\xi_\mu = g_{\mu\alpha} \xi^\alpha = g_{\mu 0}$. Since ξ_μ is a null vector then $\xi_0 = 0$. Hence we get $g_{00} = 0, g_{01} = \xi_1, g_{02} = \xi_2$. The corresponding spacetime metric can be given as $(x^\mu = (t, r, z))$

$$ds^2 = 2(\xi_\mu dx^\mu) dt + m^2 dr^2 + n^2 dz^2 + 2\ell dr dz \quad (62)$$

where ξ_1, ξ_2, m, n and ℓ are functions of r and z . Let $\ell = 0$, then $\det(g) \equiv -\Delta^2 = -(\xi_1^2 n^2 + \xi_2^2 m^2)$, and the function ψ defined in (52) takes the form $\psi = \frac{\xi_1 w_{,z} - \xi_2 w_{,r}}{\Delta}$ and

$$g_{\mu\nu} = \begin{pmatrix} 0 & \xi_1 & \xi_2 \\ \xi_1 & m^2 & 0 \\ \xi_2 & 0 & n^2 \end{pmatrix}, \quad g^{\mu\nu} = \frac{1}{\Delta^2} \begin{pmatrix} -m^2 n^2 & \xi_1 n^2 & \xi_2 m^2 \\ \xi_1 n^2 & \xi_2^2 & -\xi_1 \xi_2 \\ \xi_2 m^2 & -\xi_1 \xi_2 & \xi_1^2 \end{pmatrix} \quad (63)$$

The equation (44) reduces to

$$\xi_{1,z} - \xi_{2,r} = 2w\Delta \quad (64)$$

The function ρ can be calculated in terms of the metric functions ξ_1, ξ_2, m and n . By using a transformation $r = f(R, Z)$ and $z = g(R, Z)$ where f and g are any differentiable functions, without loosing any generality, we can let one of the functions ξ_1 or ξ_2 zero. Here in this work we will take $\xi_1 = 0$ and let $\xi_2 = q$. With these simplifications we obtain $\Delta = qm$, $q_{,r} = -2wqm$, and

$$ds^2 = 2q dz dt + m^2 dr^2 + n^2 dz^2. \quad (65)$$

where w is a nonzero constant. With such a choice we have

$$\rho = \frac{1}{m^3 q^3} [2wqm^2 n n_{,r} + nqm_{,r} n_{,r} - m^2 qm_{,zz} + m^2 m_{,z} q_{,z} - mnqn_{,rr} - mq(n_{,r})^2] \quad (66)$$

Remark 3. If $w = 0$, the Killing vector becomes hypersurface orthogonal and q becomes a constant. Then

$$\rho = \frac{1}{m^3 q^2} [nqm_{,r} n_{,r} - m^2 m_{,zz} + m^2 m_{,z} q_{,z} - mnn_{,rr} - m(n_{,r})^2]. \quad (67)$$

(5.B). Letting $q = e^y$ and $n = m$ we get $m = -\frac{1}{2w} y_{,r}$, $w \neq 0$. Here y is a function of r and z . Then

$$\rho = \frac{1}{e^{2y} y_{,r}} [-(y_{,zz} + y_{,rr}) + \frac{1}{2} ((y_{,r})^2 + (y_{,z})^2)],_r \quad (68)$$

We get a nonlinear partial differential equation for y when $\rho = \rho_0$ is a constant.

$$-(y_{,zz} + y_{,rr}) + \frac{1}{2}((y_{,r})^2 + (y_{,z})^2) = \frac{1}{2}\rho_0 e^{2y} + \rho_1 \quad (69)$$

where ρ_1 is an integration constant. An exact solution of the above equation is given as $y = -\ln[A + B(r - r_0)^2]$ where $\rho_1 = 0$ and $\rho_0 = 4AB$. After performing some scale transformations $z \rightarrow \sqrt{A}z$, $r \rightarrow \sqrt{A}r$ and letting $B = w$ and $r_0 = 0$ the metric becomes

$$ds^2 = \frac{dt dz}{\rho_0 + w r^2} + \frac{r^2}{[\rho_0 + w r^2]^2} (dr^2 + dz^2) \quad (70)$$

This metric solves Einstein field equations (61) with null fluid whose energy density $\rho = \rho_0$ is a constant. All curvature invariants are constants. For instance $R = -6w^2$, $R^{\alpha\beta} R_{\alpha\beta} = 12w^4$. Hence the spacetime is not asymptotically flat. When $\rho_0 = 0$ the solution corresponds to AdS_3 .

(5.C). When $w = 0$, the Killing vector becomes covariantly constant. By taking $m = n$ the function ρ becomes

$$\rho = -\frac{1}{mq} (m_{zz} + m_{rr}) \quad (71)$$

and the metric takes the form

$$ds^2 = 2q dz dt + m^2 (dr^2 + dz^2). \quad (72)$$

where q is a constant. Einstein tensor becomes

$$G_{\mu\nu} = \rho \xi_\mu \xi_\nu \quad (73)$$

All scalars constructed from the Ricci tensor vanish. We will not consider this case in the sequel.

6 Topologically Massive Gravity Theory

Topologically Massive Gravity (TMG) equations found by Deser, Jackiw and Templeton (DJT) [1]. Recently [3], [4] this theory was extended to the case with a cosmological constant. They are given as follows.

$$G_{\nu}^{\mu} + \frac{1}{\mu} C^{\mu}_{\nu} = \lambda \delta_{\nu}^{\mu}. \quad (74)$$

Here $G_{\mu\nu}$ and $R_{\mu\nu}$ are the Einstein and Ricci tensors respectively and C_{ν}^{μ} is the Cotton tensor which is given by

$$C^{\mu}_{\nu} = \eta^{\mu\beta\alpha} (R_{\nu\beta} - \frac{1}{4} R g_{\nu\beta})_{;\alpha}. \quad (75)$$

The constants μ and λ are respectively the DJT parameter and the cosmological constant. Solutions of this theory were studied by several authors [10]-[29].

1. When the spacetime admits a non-null Killing vector field u^{μ} , assuming that the ricci scalar R is a constant (or the Gaussian curvature, $R = 2K + 2w^2$, of the two spaces orthogonal to the Killing directions is a constant) and using the Eq. (34) we get

$$C_{\mu\nu} = -w (3w^2 + \frac{R}{2}) [g_{\mu\nu} - \frac{3}{\alpha} u_{\mu} u_{\nu}]. \quad (76)$$

Using the TMG field equations we obtain

$$\mu = 3w, \quad \lambda = -\frac{R}{6}. \quad (77)$$

which is valid for both spacelike and timelike cases. The metrics (35) with equation (36) and (37) with equation (38) are exact solutions of TMG with parameters satisfying the conditions (77).

2. If the spacetime admits a null Killing vector the Einstein tensor takes the form (61). Using the Killing equation (44) and the Einstein tensor (61) we get

$$C^{\mu}_{\nu} = (w\rho - \sigma) \xi_{\mu} \xi_{\nu}, \quad \sigma = \frac{\xi_1 \rho_{,z} - \xi_2 \rho_{,r}}{\Delta} \quad (78)$$

which leads to the following equations

$$\lambda = w^2, \quad (79)$$

$$(\mu + w)\rho = \sigma = \frac{\xi_1 \rho_{,z} - \xi_2 \rho_{,r}}{\Delta}. \quad (80)$$

In addition to the Killing equation (64) these equations constitute the field equations to be solved for the DJT Theory. With the simplification done in **5.A** of the last section we get

$$\rho = \rho_0 q^{\frac{\mu+w}{2w}}, \quad m = -\frac{q_{,r}}{2wq} \quad (81)$$

where ρ_0 is an arbitrary constant. As far as the solutions are concerned we have the following three classes:

a). We obtain a simple solution of (80) when $\mu = -w$ which leads to $\rho = \rho_0$ constant. Then using the simplifications done in **5.B** of last section ($\xi_1 = 0$ and $\xi_2 = q = e^y$, $n = m = -\frac{1}{2w} y_{,r}$) we get all metric functions related to the function y which satisfies the differential equation (69). An exact solution and metric of the spacetime is given in (70) of **5.B** part of the last section.

b). A more general solution is obtained when $\mu + 3w = 0$ where the function y satisfies the equation

$$-(y_{,zz} + y_{,rr}) + \frac{1}{2} ((y_{,r})^2 + (y_{,z})^2) = \rho_0 y + \rho_2 \quad (82)$$

where ρ_2 is an integration constant. A solution of this equation is $y = \frac{\rho_0}{2} r^2$, $\rho_2 = -\rho_0$. Then the metric becomes

$$ds^2 = e^{\frac{\rho_0}{2} r^2} dt dz + \frac{\rho_0^2 r^2}{4 w^2} [dr^2 + dz^2] \quad (83)$$

Here $\rho_0 \neq 0$.

c). The case when $\mu + 5w \neq 0$. Function y satisfies the equation

$$-(y_{,zz} + y_{,rr}) + \frac{1}{2} ((y_{,r})^2 + (y_{,z})^2) = \frac{\rho_0}{\epsilon} e^{\epsilon y} + \rho_3 \quad (84)$$

where ρ_3 is an integration constant and $\epsilon = \frac{\mu+5w}{2w}$. The circularly symmetric metric has the form

$$ds^2 = e^y dt dz + \frac{1}{4 w^2} e^{\epsilon y} \left[-\frac{2\rho_0}{\epsilon(\epsilon-1)} + 2\rho_3 e^{-\epsilon y} + \rho_4 e^{(1-\epsilon)y} \right] (dr^2 + dz^2) \quad (85)$$

where the function $y = y(r)$ satisfies the equation

$$y_r = \pm e^{\frac{\epsilon}{2}y} \sqrt{-\frac{2\rho_0}{\epsilon(\epsilon-1)} + 2\rho_3 e^{-\epsilon y} + \rho_4 e^{(1-\epsilon)y}} \quad (86)$$

Here ρ_4 is also an integration constant. All other cases will be considered later.

Remark 4. In the case of non-null Killing vector field when $w = 0$ Killing vector becomes hypersurface orthogonal and the Cotton tensor vanishes. TMG field equations with hypersurface orthogonal Killing vectors were previously considered. With a hypersurface orthogonal Killing vector Aliev and Nutku showed that TMG allow only flat solution, [40]. Recently Deser and Franklin in [41] considered circularly symmetric spacetime in TMG. They showed that such symmetry (existence of the hypersurface orthogonal space-like Killing vector $\frac{\partial}{\partial\theta}$) breaks TMG with cosmological constant down to vanishing of Cotton and Einstein sectors separately.

7 A New Massive Gravity in Three Dimensions

Recently a new, parity-preserving theory is introduced by Bergshoeff-Hohm-Townsend (BHT) [5] in three dimensions which is equivalent to Pauli-Fierz massive field theory at the linearized level. Originally this theory, known as NMG does not contain a cosmological term in the action. Adding a cosmological constant λ the new massive gravity field equations (CNMG) are given as

$$2m_0^2 G_{\mu\nu} + K_{\mu\nu} + \lambda g_{\mu\nu} = 0 \quad (87)$$

where

$$\begin{aligned} K_{\mu\nu} = & 2\nabla^2 R_{\mu\nu} - \frac{1}{2}(R_{;\mu\nu} + g_{\mu\nu} \nabla^2 R) - 8R^\rho{}_\mu R_{\nu\rho} \\ & + \frac{9}{2}R R_{\mu\nu} + [3R^{\alpha\beta} R_{\alpha\beta} - \frac{13}{8}R^2] g_{\mu\nu} \end{aligned} \quad (88)$$

where m_0 is relative mass parameter and ∇^2 is the Laplace-Beltrami operator. Here we used the mass parameter as m_0 not to confuse with the metric

function m . Solutions of these equations have been recently studied in [15], [6]

1. When the spacetime admits a non-null Killing vector field, assuming that the ricci scalar R is a constant and using the equation (34) we get $R = 2K + 2w^2$ and

$$\nabla^2 R_{\mu\nu} = 2w^2 (3w^2 + \frac{R}{2}) [g_{\mu\nu} - \frac{3}{\alpha} u_\mu u_\nu], \quad (89)$$

$$R^{\mu\alpha} R_{\nu\alpha} = \frac{1}{\alpha} (3w^2 + \frac{R}{2}) (w^2 - \frac{R}{2}) u^\mu u_\nu + (w^2 + \frac{R}{2})^2 \delta^\mu{}_\nu, \quad (90)$$

$$K_{\mu\nu} = -\frac{1}{\alpha} (3w^2 + \frac{R}{2}) (20w^2 + \frac{R}{2}) u_\mu u_\nu + (22w^4 + \frac{9}{2} w^2 R + \frac{R^2}{8}) g_{\mu\nu}. \quad (91)$$

Then using the field equations of NMG gravity we obtain $(3w^2 + \frac{R}{2} \neq 0)$

$$m_0^2 = -10w^2 - \frac{R}{4}, \quad \lambda = -2w^4 - \frac{R^2}{8} - 4w^2 R. \quad (92)$$

The metrics (35) with equation (36) and (37) with equation (38) are exact solutions of NMG with parameters satisfying the conditions (92). As an example the metric found in [10] as an exact solution of TMG is also an exact solution of the NMG with parameters satisfying the conditions (92).

If $3w^2 + \frac{R}{2} = 0$ then we obtain

$$\lambda + 2m_0^2 w^2 + \frac{1}{2} w^4 = 0 \quad (93)$$

2. When the spacetime admits a null Killing vector field, using Eq.(44) and the Einstein tensor (61) we obtain $R = -6w^2$ and

$$\nabla^2 R_{\mu\nu} = (\nabla^2 \rho + 4w\sigma + 6w^2\rho) \xi_\mu \xi_\nu, \quad (94)$$

$$R^\rho{}_\mu R_{\rho\nu} = -4w^2 \rho \xi_\mu \xi_\nu + 4w^4 g_{\mu\nu}, \quad (95)$$

$$K_{\mu\nu} = [2 \nabla^2 \rho + 8w\sigma + 17w^2\rho] \xi_\mu \xi_\nu - \frac{1}{2} w^4 g_{\mu\nu}. \quad (96)$$

These equations lead to the following results

$$2 \nabla^2 \rho + (2 m_0^2 + 17 w^2) \rho + 8 w \sigma = 0, \quad (97)$$

$$4 m_0^2 w^2 - w^4 + 2 \lambda = 0. \quad (98)$$

The full massive gravity field equations reduce to (97), (98) and the equation (44). A circularly symmetric solution of this equation is given by $\rho = \rho_0 e^{ky}$ where ρ_0 is a constant and k satisfies the quadratic equation $8k^2 + 24k + 35/2 - \frac{\lambda}{w^4} = 0$. The circularly symmetric metric becomes

$$ds^2 = e^{2y} dt dz + \frac{1}{4 w^2} \left[\frac{2 \rho_0}{k-1} e^{ky} + \rho_1 e^y \right] (dr^2 + dz^2) \quad (99)$$

Here y satisfies a similar equation like in (86)

$$y_{,r} = \pm \sqrt{\frac{2 \rho_0}{k-1} e^{ky} + \rho_1 e^y} \quad (100)$$

where ρ_1 is an integration constant. For each value of k (k_1 and k_2) we have two different metrics.

BHT introduced also a more general model [5] which also includes the topologically massive gravity as a special case.

$$\lambda m_0^2 g_{\mu\nu} + \alpha G_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} + \frac{\beta}{2 m_0^2} K_{\mu\nu} = 0, \quad (101)$$

where λ , α and β are dimensionless parameters. For $m_0 \rightarrow \infty$ and for fixed μ generalized BHT equations reduce to the DJT equations (74). The solutions of these equations are given as follows:

1. Non-null case: $(3w^2 + \frac{R}{2} \neq 0)$

$$-\alpha + \frac{3w}{\mu} - \frac{\beta}{2 m_0^2} (20w^2 + R/2) = 0, \quad (102)$$

$$\lambda m_0^2 + \alpha w^2 - \frac{w}{\mu} (3w^2 + \frac{R}{2}) + \frac{\beta}{2 m_0^2} (22w^4 + \frac{9}{2} w^2 R + \frac{R^2}{8}) = 0 \quad (103)$$

If $3w^2 + \frac{R}{2} = 0$ then

$$\lambda m_0^2 + \alpha w^2 + \frac{\beta}{4 m_0^2} w^4 = 0 \quad (104)$$

2. Null case:

$$\frac{\beta}{m_0^2} \nabla^2 \rho + \left(\alpha + \frac{w}{\mu} + \frac{17w^2\beta}{2m_0^2} \right) \rho + \left(-\frac{1}{\mu} + \frac{4\beta w}{m_0^2} \right) \sigma = 0, \quad (105)$$

$$\lambda m_0^2 + \alpha w^2 - \frac{\beta}{4m_0^2} w^4 = 0 \quad (106)$$

In a similar fashion letting $\rho = \rho_0 e^{ky}$ we get a quadratic equation for k

$$4w^2 \frac{\beta}{m_0^2} k(k+1) + 2\left(-\frac{1}{\mu} + \frac{4\beta w}{m_0^2}\right) w k + \left(\alpha + \frac{w}{\mu} + \frac{17w^2\beta}{2m_0^2}\right) = 0 \quad (107)$$

In order that the roots to exists the following condition should be satisfied

$$\left(-\frac{1}{\mu} + \frac{4\beta w}{m_0^2} + 2\frac{\beta^2}{m_0^2}\right)^2 - 4\frac{\beta}{m_0^2} \left(\alpha + \frac{w}{\mu} + \frac{17w^2\beta}{2m_0^2}\right) w^2 \geq 0 \quad (108)$$

The metric function y satisfies exactly the same equation (100) but k solves the quadratic equation (107) in this case. The circularly symmetric metric is of the form given (99) where $y(r)$ satisfies (100).

Both for null and non-null cases when the scalars of the Killing vector fields are constants the we have a more general result. The following conjecture implies that the corresponding metrics solve all higher derivative gravitational field equations in three dimensions.

Conjecture. *Let a three dimensional spacetime admit a Killing vector field u_μ (non-null or null) with constant scalars. Let the Gaussian curvature K of the two dimensional spaces be constant for the case of non-null vector fields where corresponding Einstein tensors are respectively given in (39) and (61). Then any symmetric second rank covariant tensor constructed from the ricci tensor by covariant differentiation and by contraction is the linear sum of $u_\mu u_\nu$ and the metric tensor $g_{\mu\nu}$*

8 Conclusion

In this work we first studied the Killing vector fields in a three dimensional spacetime geometry. We showed that, independent of the type of the Killing

vector fields, the ricci tensor can be determined in terms of the Killing vector fields and their scalars. Usually components of this tensor are calculated in terms of the components of metric tensor in a given coordinate system. In three dimensions, when the geometry admits at least one Killing vector field we don't have to follow such a direction to determine the components of the ricci tensor. Using this property in each case (when the Killing vector field is timelike, spacelike or null) we first presented solutions of Einstein field equations with sources. Then by using special cases, when the scalars of the Killing vector fields are constants, we gave solutions of the field equations of the Topologically Massive Gravity and New Massive Gravity theories . Some of the solutions of the Topologically Massive Gravity field equations we obtained in this work may already be known. Our basic purpose in this work is to present a new method to solve higher derivative gravity theories rather finding specific solutions. We conjecture at the end that, for the three type of Killing vector fields with constant scalars our method solves all higher derivative theories in three dimensions.

This work is partially supported by the Scientific and Technological Research Council of Turkey (TUBITAK) and Turkish Academy of Sciences (TUBA).

References

- [1] S. Deser, R. Jackiw, and S. Templeton, *Phys. Rev. Lett.* **48**, 975 (1982)
- [2] S. Deser, R. Jackiw, and S. Templeton, *Annals. Phys., NY* **140**, 372 (1982)[Erratum-ibid. **185**, 406 (1988)].
- [3] S. Carlip, S. Deser, A. Waldraon, and D. K. Wise, *Class. Quantum Grav.*, **26**, 075008 (2009); *Phys. Lett.*, **B666**, 272 (2008).

- [4] W. Ki. Song and A. Strominger, J. High Energy Phys. **0804**, 082 (2008).
- [5] E. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. Lett., **102**, 201301 (2009).
- [6] E. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. D79, 24042 (2009).
- [7] E. Bergshoeff, O. Hohm and P. K. Townsend, *On massive gravitons in 2+1 dimensions*, arXiv. 0912.2944 [hep-th], in the proceedings of the Spanish Relativity Meeting ERE2009, Bilbao;
- [8] S. Deser, Phys. Rev. Lett. **103**, 101302 (2009).
- [9] B. Tekin and I. Güllü, Phys. Rev. D80,064033 (2009).
- [10] M. Gürses, Class. Quantum Grav. **11**, 2585 (1994).
- [11] G. Clement, Phys. Rev.**49**, 5131 (1994).
- [12] G. Clement, Class. Quantum Grav. **11**,L115 (1994).
- [13] K.A. Moussa, G. Clement, and C. Leygnac, Class. Quantum Grav. **20**,L277-L283(2003).
- [14] K.A. Moussa, G. Clement, H. Guennoune and C. Leygnac, Phys.Rev. **D78**, 064065 (2008).
- [15] G. Clement, Class. Quantum Grav. **26**, 105015 (2009).
- [16] G. Clement, Class. Quantum Grav. **26**, 165002 (2009).
- [17] R.S. Palais, Commun. Math. Phys., **69**, 19(1979).
- [18] M.E. Fels and C.G. Torre, Class. Quantum Grav. **19**, 641(2002).
- [19] S. Deser and B. Tekin, Class. Quantum Grav.**20**, 4877 (2003).
- [20] D.D.K. Chow, C.N. Pope and E. Sezgin, Class. Quant. Grav. **27**, 105001(2010).

- [21] D. Anninos, W. Li, W. Song, and A. Strominger, JHEP, **0903**, 130 (2009).
- [22] S. Deser, R. Jackiw, and S. Y. Pi, *Cotton Blend Gravity pp-Waves*, Dedicated to Andrzej Staruszkiewicz, a pioneer in d=3 gravity, on his 65th birthday, ActaPhys.Polon. **B36**, 27-34 (2005). (arXiv:gr-qc/0409011).
- [23] E. Ayon-Beato and M. Hassaine, Annals Phys. **317**, 175 (2005).
- [24] E. Ayon-Beato and M. Hassaine, Phys. Rev. **D73**, 104001 (2006).
- [25] D.D.K. Chow, C.N. Pope and E. Sezgin, Class. Quant. Grav. **27**, 105002(2010).
- [26] M. Gürses, A. Karasu and Ö. Sarioğlu, Class. Quantum Grav. **22**, 1527-1543 (2005).
- [27] M. Gürses and Ö. Sarioğlu, Class. Quantum Grav. **22**, 4699(2005).
- [28] R.J. Gleiser, M. Gürses, A. Karasu and Ö. Sarioğlu, Class. Quantum Grav. **23**, 2653 (2006).
- [29] M. Gürses, Gen. Relativ. Gravit. **42**, 1413 (2010).
- [30] J.D. Barrow, D.J. Shaw, and C.G. Tsagas, Class. Quantum Grav. **23**, 5291-5322 (2006).
- [31] Y. Nutku, Class. Quantum. Grav. **10**, 2657 (1993).
- [32] I. Vuorio, Phys. Lett. **B163**, 91 (1985); R. Percacci, P. Sodano, and I. Vuorio, Ann. Phys. NY, **176**, 344 (1987).
- [33] E. Ayon-Beato, G. Giribet and M. Hassaine, JHEP **0905**, 029 (2009).
- [34] S.W. Hawking and G.F.R. Ellis, **The large Scale structure of spacetime**, (Cambridge University Press 1973).
- [35] B. Carter, J. Math. Phys. **10**, 70 (1969).
- [36] M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. **D 48**, 1506 (1993).

- [37] L.P. Eisenhart, **Riemannian Geometry**, (Princeton University Press, Fifth printing 1964).
- [38] G. S. Hall, J. Math. Phys. **31**, 1198 (1990).
- [39] J. Carot, Class. Quantum. Grav. **17**, 2675 (2000).
- [40] Aliev and Y. Nutku, Class. Quantum Grav. **13** , L29-L32 (1996).
- [41] S. Deser and J. Franklin, Class. Quantum. Grav. **27**, 1007002 (2010).